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Helmholtz resonance of harbours

By JOHN W. MILES AND Y. K. LEE†

Institute of Geophysics and Planetary Physics,
University of California, La Jolla

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The resonant response of a harbour H of depth scale d and area A to excitation of frequency ω through a mouth M of width a is calculated in the joint limit a^2/A, $\omega^2 A/gd \downarrow 0$. The results are relevant to the tsunami response of narrow-mouthed harbours. It is assumed that an adequate approximation to the radiation impedance of the external domain is available (Miles 1972). The boundary-value problem for H is reduced to the solution of $\nabla . (\hbar\nabla\phi) = -1/A$, where $\hbar$ is the relative depth, the normal derivative of ϕ is prescribed in M and vanishes elsewhere on the boundary of H, and the spatial mean of ϕ must vanish. The kinetic energy in H is proportional to an inertial parameter $\mathscr{M}$ that is a quadratic functional of ϕ. It is demonstrated that decreasing/increasing $\hbar$ increases/decreases $\mathscr{M}$. Explicit lower bounds to $\mathscr{M}$ are developed for both uniform and variable depth. The results are extended to coupled basins (inner and outer harbours). Several examples are considered, including a model of Long Beach Harbor, for which the calculated resonant frequency of the dominant mode is within 1 % of the measured value. The effects of entry-separation and bottom-friction losses are considered; the latter are typically negligible, whereas the former may be comparable with, or dominate, radiation losses.

1. Introduction

We consider small disturbances within a harbour or bay H (see figure 1) that opens to an external sea E through a mouth M under the geometrical restrictions

$$\alpha \equiv a/R \ll 1, \quad d = O(R), \qquad (1.1a, b)$$

and the dynamical restriction

$$\epsilon \equiv \omega^2 A/gd = \pi(kR)^2 \ll 1, \qquad (1.2)$$

where a is the width of M, $A \equiv \pi R^2$ is the free-surface area, d is a characteristic depth (subsequently taken to be the mean depth of M), ω is the angular frequency, $c^2 = gd$ is the square of the wave speed based on d, and $k = \omega/c$ is the corresponding wavenumber. We assume simple harmonic motion, such that the horizontal velocity $\hat{\mathbf{u}}$ and the free-surface displacement $\hat{\zeta}$ at the point $\mathbf{r}$ are given by

$$\{\hat{\mathbf{u}}(\mathbf{r}, t), \hat{\zeta}(\mathbf{r}, t)\} = \mathscr{R}[\{\mathbf{u}(\mathbf{r}), \zeta(\mathbf{r})\} e^{i\omega t}], \qquad (1.3)$$

where $\mathscr{R}$ implies *the real part of*, i is the imaginary unit, and $\mathbf{u} = \{u, v\}$ and ζ are the complex amplitudes of $\hat{\mathbf{u}}$ and $\hat{\zeta}$ (we omit the modifier *complex amplitude of*

† Present address: Tetra Tech, Inc., Pasadena, California.

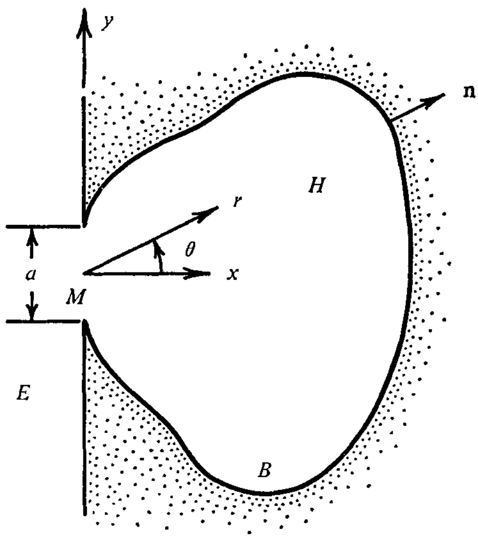

FIGURE 1. Schematic diagram of harbour (H) opening into semi-infinite exterior domain (E) through mouth (M); r and θ are plane polar co-ordinates with origin at the midpoint of M, and $\mathbf{n}$ is the unit normal to the lateral boundary (B).

throughout the subsequent exposition and refer to $\mathbf{u}$ and ζ simply as the velocity and displacement). Any slowly varying motion may be treated by spectral superposition provided that the disturbance is small and (1.2) is satisfied by the highest significant frequency.

The joint restrictions (1.1) and (1.2) imply that the response of H to excitation in E is dominated by the Helmholtz mode, in which ζ is approximately uniform over H except in a small neighbourhood of M and $\mathbf{u}$ is strongly localized in M. Previous studies of this mode have been restricted to constant depth, and analytical results are available only for simple shapes (Miles & Munk 1961; Carrier, Shaw & Miyata 1971*a*, *b*; Lee 1971; Miles 1971). The present study develops analytical approximations for irregularly shaped harbours of variable depth. [The description *Helmholtz mode* appears to have been introduced in this context by Miles & Munk (1961); the synonym *pumping mode* also is used (Lee 1971).]

Let $\langle\zeta\rangle$ be the mean displacement in H, where, here and until §7, $\langle\,\rangle$ implies a spatial average over H, and let I be the integrated volume flux through M into H; then continuity implies that I must be equal to the rate of change of volume in H:

$$I \equiv \int_M uh\,dy = \int_H (i\omega\zeta)\,dA = i\omega A\langle\zeta\rangle, \tag{1.4}$$

where h is the local depth (note that, by definition, M is in the plane $x = 0$; see figure 1). The hypothesis that ζ is approximately uniform over H implies that the temporal mean of the potential energy in H may be approximated according to (after invoking $\frac{1}{2}$ for the temporal mean of $\cos^2\omega t$)

$$U = \tfrac{1}{4}\rho g\int_H |\zeta|^2 dA \downarrow \tfrac{1}{4}\rho gA|\langle\zeta\rangle|^2 \quad (\epsilon\downarrow 0). \tag{1.5}$$

Dimensional considerations suggest that the temporal mean of the kinetic energy in H may be posed in the form

$$K = \tfrac{1}{4}\rho(\mathscr{M}/d)|I|^2, \tag{1.6}$$

where $\mathscr{M}$ is a dimensionless inertial parameter. The calculation of $\mathscr{M}$ requires the solution of a boundary-value problem, which is formulated in §2 and leads to the representation of $\mathscr{M}$ as a quadratic functional of a potential that satisfies a generalization of Poisson's equation.† Various bounds to this representation are constructed in §3 and applied in §§4 and 5. Coupled basins that are connected to an external sea to form inner and outer harbours are considered in §6.

We remark that, if H is reflected in the plane of M to obtain a pair of identical basins connected through M, the potential and kinetic energies in the dominant mode of the coupled basins are $2U$ and $2K$. Equating (1.5) and (1.6) and invoking (1.4) then yields the resonant period

$$T_0 = 2\pi(\mathscr{M}A/gd)^{\frac{1}{2}}, \tag{1.7}$$

which suggests a possible experimental determination of $\mathscr{M}$. If the basins are connected by a short channel of length l, $\mathscr{M}$ in (1.7) is replaced by $\mathscr{M}+\frac{1}{2}\mathscr{M}_l$, where $\mathscr{M}_l$ is given by (1.12) below. The resulting oscillations resemble those in a U-tube and were considered by Neumann (1943) on the implicit hypothesis $\mathscr{M}_l \gg \mathscr{M}$.

The calculation of $\langle\zeta\rangle$ is facilitated by the equivalent circuit shown in figure 2(*a*) (Miles 1971, hereinafter referenced by M71, followed by the appropriate section number), in which the input voltage $V_E \equiv \zeta_E$ is the displacement that would exist just outside M if H were closed (the variation of ζ across M is negligible by hypothesis), V_M is a measure of the actual displacement in M, and $V_H \equiv \langle\zeta\rangle$; the current I is defined by (1.4); $Z_E \equiv R_E + i\omega L_E$ is the radiation impedance of E with respect to M;

$$C_H = A, \quad L_H = \mathscr{M}/gd \tag{1.8a,b}$$

are the equivalent capacitance and inductance of H;‡

$$Z_H = (i\omega C_H)^{-1} + i\omega L_H = (i\omega A)^{-1}(1-\epsilon\mathscr{M}) \tag{1.9a,b}$$

is the impedance of H with respect to M. The normalizations of equivalent voltage and equivalent current are such that the electrostatic energy stored in C_H and the magnetic energies stored in L_H and L_E are equal to U, K and the stored energy in E, respectively, divided by the specific weight ρg; similarly, the ohmic dissipation rate in R_E is equal to the radiated power divided by ρg.

The explicit calculation of the radiation impedance Z_E is possible only for relatively special configurations, such as an ocean of constant depth (M71 §3). In general, the external domain is characterized by both a continuous spectrum

† Helmholtz resonance of a harbour of constant depth is analogous to the corresponding two-dimensional acoustical problem, and the parameter $d/\mathscr{M}$ is analogous to the 'conductivity' of the acoustical resonator. The calculation of $\mathscr{M}$ is much simpler for a three-dimensional resonator (Rayleigh 1896, §§304–308), in which the disturbance radiated from M vanishes inversely with the distance from M.

‡ The parameters ϵ, C_H, L_H and $\mathscr{M}$ are denoted by κ, C_0, L_H and $\Lambda_H^{(0)}$ in M71.

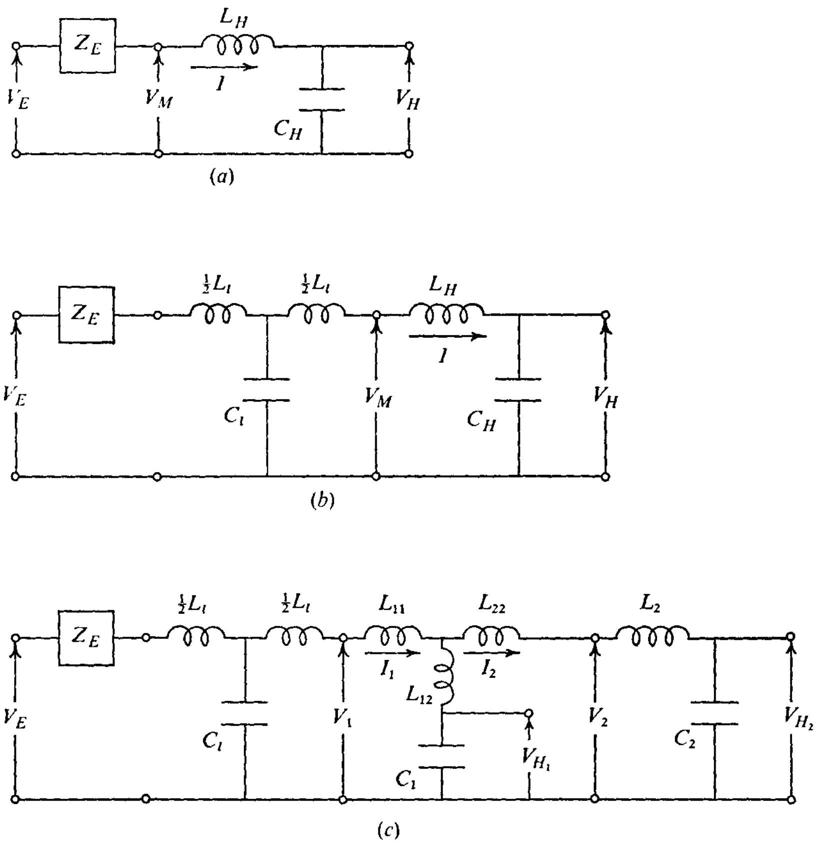

FIGURE 2. Equivalent circuit (a) for harbour and exterior domain, (b) with incorporation of entry channel and (c) extended to coupled basins (see § 6).

associated with the abyss (assumed to be semi-infinite in extent) and a discrete spectrum associated with the continental-shelf modes, and Z_E exhibits complicated variations with frequency if, as may be true for tsunamis, the wavelength of the disturbance in E (which may be much longer than the wavelength in H owing to the greater depth) is comparable with the width of the shelf. On the other hand, the greater depth in E reduces the loading on H, and a rough approximation to Z_E is likely to be adequate for the calculation of the response of H. In particular, it appears that an adequate approximation for the calculation of Helmholtz resonance is given by (Miles 1972, wherein $\mathscr{r}$ is designated by ρ)

$$Z_E = (\omega/gd_E)\,[\mathscr{r} + i\{0{\cdot}660 - (2\pi)^{-1}\log(\alpha^2\epsilon d/d_E)\}] \tag{1.10a}$$

$$\equiv R_E + i\omega L_E \equiv (\omega/gd)\,(\mathscr{R}_E + i\mathscr{M}_E), \tag{1.10b}$$

where d_E is the depth just outside M and $\mathscr{r}$ is an oscillatory function of the product of shelf wavenumber and shelf width that tends asymptotically to $\frac{1}{2}$.

The existence of even a short entry channel between E and H can have a significant effect on the response of H and, especially, Helmholtz resonance (Carrier *et al.* 1971*a*, *b*). A channel of length l, uniform breadth b (not necessarily equal to a) and uniform depth d may be represented by a four-terminal network inserted between Z_E and Z_H, as shown in figure 2(b), wherein (M71 §5)

$$C_l = bl, \quad L_l = \mathscr{M}_l/gd, \quad \mathscr{M}_l = l/b \qquad (1.11\,a, b, c)$$

in the limit $\epsilon \downarrow 0$ with $l = O(R)$. A straightforward extension of the analysis of M71 §5, based on the equations of quasi-one-dimensional channel motion (Lamb 1932, §185), implies that, if the dimensions of the channel are non-uniform, bl in (1.11a) should be replaced by the corresponding area and (1.11c) by

$$\mathscr{M}_l = d\int_0^l \{dx/S(x)\}, \qquad (1.12)$$

where $S(x)$ is the cross-sectional area of the channel.

The amplification factor for the equivalent circuit of figure 2(b) is given by

$$\langle\zeta\rangle/\zeta_E = V_H/V_E = (1+\alpha)\,[1-\epsilon\{\mathscr{M}+\mathscr{M}_l+(1+\alpha)\,\mathscr{M}_E\}+i\epsilon(1+\alpha)\,\mathscr{R}_E]^{-1}, \quad (1.13)$$

where α is the ratio of the free-surface area of the channel (if any) to that of the basin. The resonant frequency $\omega = \omega_0$ is determined by

$$\epsilon\{\mathscr{M}+\mathscr{M}_l+(1+\alpha)\,\mathscr{M}_E\} = 1 \quad (\omega = \omega_0). \qquad (1.14)$$

The dependence of $\mathscr{M}_E$ on ϵ prevents an explicit solution of (1.14), but the solution is readily obtained by iteration. The corresponding period is given by

$$T_0 = 2\pi(A/gd)^{\frac{1}{2}}\{\mathscr{M}+\mathscr{M}_l+(1+\alpha)\,\mathscr{M}_E\}^{\frac{1}{2}}, \qquad (1.15)$$

in which $\mathscr{M}_E$ may be neglected for sufficiently small d/d_E. The Q of the series-resonant circuit is given by

$$1/Q_0 = \epsilon_0(1+\alpha)\,\mathscr{R}_E = (1+\alpha)\,\{\mathscr{M}+\mathscr{M}_l+(1+\alpha)\,\mathscr{M}_E\}^{-1}\,\mathscr{R}_E. \qquad (1.16\,a, b)$$

We emphasize that the r.m.s. displacement in H for a prescribed random input with a power spectrum that comprises ω_0 is proportional to $Q_0^{\frac{1}{2}}$, rather than to Q_0; see M71 §4.

The preceding discussion neglects frictional losses in boundary layers and in the eddying flow associated with separation at the mouth. The separation losses, which may be comparable with, or even dominate, radiation losses at resonance, are considered in §7† and may be represented by a resistor R_S in series with L_H (separation may occur at both ends of an entry channel, but the losses may be represented by a single resistor if $\alpha \ll 1$). Boundary-layer losses in the harbour may be represented by an additional series resistor R_B; they are typically small compared with the separation losses (see §8). Boundary-layer losses in a narrow entry channel,‡ which may be much larger than those in the harbour, may be represented by an additional resistor R_l in series with L_l if $\alpha \ll 1$. These resistors, in contrast to R_E, are necessarily nonlinear (we exclude the unrealistic possibility of laminar boundary layers), but it suffices for the calculation of resonant re-

† Entry separation, including harmonic distortion, has also been considered in a recent study by Mei & Ünlüata (1974, private communication). Their results for the Helmholtz mode are similar to those in §7.

‡ Shaw & Lai (1974, private communication) also have considered these losses.

sponse to neglect the harmonic distortion that accompanies nonlinearity and to define

$$R \equiv 2D/|I|^2, \tag{1.17}$$

where D is the (temporal) mean dissipation rate associated with the flow of I through R. In keeping with this approach, we neglect frictional losses in §§2–6, which proceed rationally from the hypothesis of an ideal fluid, and then calculate the losses by *ad hoc* methods in §§7 and 8. The corresponding Q may be estimated by replacing R_E by the sum of R_E, R_S (separation losses) and R_B (boundary-layer losses). If the effective value of R is sufficiently large, as for a breakwater with a narrow opening (see §7), the resonant response may be almost completely suppressed, in which case Q has only qualitative significance.

2. Interior boundary-value problem

The restrictions (1.1b) and (1.2), together with the assumption of small displacements, imply that $\mathbf{u}$ and ζ satisfy the linearized shallow-water equations (Lamb 1932, §§189, 193)

$$i\omega\mathbf{u} = -g\nabla\zeta \tag{2.1}$$

and

$$\nabla.(h\mathbf{u}) + i\omega\zeta = 0, \tag{2.2}$$

where ∇ is the two-dimensional gradient operator and h is the local depth. The boundary-value problem for H, which is a sub-problem of the complete problem for $H+E$, requires the solution of (2.1) and (2.2) subject to the boundary conditions

$$h(\mathbf{n}.\mathbf{u}) = 0 \quad \text{on} \quad B \tag{2.3a}$$

and

$$hu = q(y) \quad \text{on} \quad M, \tag{2.3b}$$

where $\mathbf{n}$ is the outwardly directed normal, B and M are the closed and open portions of the lateral boundary of H, and $q(y)$ is the prescribed (or assumed) flow per unit width into M.

The complete problem requires the solution of the exterior boundary-value problem for ζ_E and the solution of the integral equation that is obtained by matching the exterior and interior solutions across M. The explicit solution of this integral equation is circumvented by posing q in the form

$$q(y) = If(y), \quad \int_M f(y)\,dy = 1, \tag{2.4a,b}$$

and estimating $f(y)$ subject to (2.4b); see discussion in M71 §2. We recall that I is related to $\langle\zeta\rangle$, the mean value of ζ over H, by (1.4).

Invoking the hypothesis that ζ is approximately uniform in the Helmholtz mode, we pose the solution of (2.1) and (2.2) in the form

$$\zeta(\mathbf{r}) = \langle\zeta\rangle\{1+\epsilon\phi(\mathbf{r})\}, \quad \mathbf{u}(\mathbf{r}) = (I/d)\,\nabla\phi(\mathbf{r}), \tag{2.5a,b}$$

where ϵ is defined by (1.2). Substituting (2.5) into (2.2) and (2.3), invoking (1.4) and (2.4), and letting $\epsilon \downarrow 0$ with $\phi = O(1)$†, we obtain

$$\nabla.(\hbar\nabla\phi) = -1/A \quad (\hbar \equiv h/d) \tag{2.6}$$

and
$$\hbar\phi_n = \begin{cases} 0 & \text{on} \quad B \\ -f(y) & \text{on} \quad M \end{cases} \quad (\phi_n \equiv \mathbf{n}.\nabla\phi), \tag{2.7a, b}$$

where $\hbar$ is the dimensionless depth. Averaging (2.5a) over H yields the constraint

$$\langle\phi\rangle = 0, \tag{2.8}$$

which renders the solution of (2.6) and (2.7) unique. We remark that ϕ is real.

The temporal mean kinetic energy in H is given by

$$K = \tfrac{1}{4}\rho\int_H |\mathbf{u}|^2 h\,dA = \tfrac{1}{4}\rho(\mathscr{M}/d)\,|I|^2, \tag{2.9}$$

where
$$\mathscr{M} = \int_H \hbar(\nabla\phi)^2 dA \equiv \mathfrak{M}_H(\phi) \tag{2.10}$$

is defined as in (1.6). Invoking Green's theorem and (2.6)–(2.8) yields the alternative representation

$$\mathscr{M} = -\int_M \phi f dy \equiv \mathfrak{M}_M(\phi). \tag{2.11}$$

We emphasize that (2.10) follows directly from (2.9) and provides the proper starting point for approximations to $\mathscr{M}$, whereas $\mathscr{M}$ is given by (2.11) only if (2.6)–(2.8) are satisfied.

3. Bounds for $\mathscr{M}$

The parameter $\mathscr{M}$, as determined by (2.6)–(2.8) and (2.10), has several classical analogues (cf. Rayleigh 1896, §§304ff) that suggest the following bounds (these bounds might be surmised from Rayleigh's principle, but direct derivations appear to be desirable).

We first note that if ϕ is the solution of (2.6)–(2.8) and if ψ, and hence also $\phi+\psi$, is any function that satisfies

$$\langle\psi\rangle = 0 \tag{3.1}$$

and for which $\mathfrak{M}_H(\psi)$ and $\mathfrak{M}_M(\psi)$ exist, then:

$$\int_H \hbar\nabla\phi.\nabla\psi dA = -\int_H \psi\nabla.(\hbar\nabla\phi)\,dA + \int_{B+M} \hbar\psi\phi_n dl \tag{3.2a}$$

$$= -\int_M \psi f dy = \mathfrak{M}_M(\psi) \tag{3.2b}$$

by virtue of Green's theorem (dl is positive counterclockwise), (2.6), (2.7) and (3.1);

$$\mathfrak{M}_H(\phi+\psi) = \mathscr{M} + \mathfrak{M}_H(\psi) + 2\mathfrak{M}_M(\psi) \tag{3.3}$$

† The approximation is not uniformly valid in $r = O(a)$. This presents no difficulty herein, but it might be either necessary or expedient to pose higher approximations in the form of matched asymptotic expansions.

by virtue of (2.10) and (3.2);

$$\mathfrak{M}_M(\phi+\psi) = \mathscr{M}+\mathfrak{M}_M(\psi) \tag{3.4}$$

by virtue of (2.11). Introducing the functional

$$\mathfrak{M}_<(\psi) \equiv 2\mathfrak{M}_M(\psi)-\mathfrak{M}_H(\psi) \tag{3.5}$$

and invoking (3.3) and (3.4) then yields

$$\mathfrak{M}_<(\phi+\psi) = \mathscr{M}-\mathfrak{M}_H(\psi) \leqslant \mathscr{M} \tag{3.6}$$

for any approximation that satisfies the constraint (3.1).

The lower bound (3.6) is useful only if the error term $\mathfrak{M}_H(\psi)$, which is quadratic in the error ψ, is small. The fact that the kinetic energy is concentrated in M suggests that a good approximation may be obtained by satisfying (2.6), (2.7b) and (2.8) and relaxing (2.7a). Let $\hat{\phi}$ be such an approximation; then Green's theorem implies

$$\mathfrak{M}_H(\hat{\phi}) \equiv \int_H \hbar(\nabla\hat{\phi})^2 dA = \int_{B+M} \hbar\hat{\phi}\hat{\phi}_n dl = \mathscr{M}_M-\mathscr{M}_B, \tag{3.7}$$

where
$$\mathscr{M}_M = \mathfrak{M}_M(\hat{\phi}), \quad \mathscr{M}_B = -\int_B \hbar\hat{\phi}\hat{\phi}_n dl \equiv \mathfrak{M}_B(\hat{\phi}). \tag{3.8a,b}$$

Setting $\phi+\psi = \hat{\phi}$ in (3.6) and invoking (3.7) and (3.8a) yields

$$\mathfrak{M}_<(\hat{\phi}) = \mathscr{M}_M+\mathscr{M}_B \leqslant \mathscr{M}. \tag{3.9}$$

We obtain a somewhat sharper lower bound,

$$\mathscr{M} \geqslant \mathscr{M}_M^2(\mathscr{M}_M-\mathscr{M}_B)^{-1} \geqslant \mathscr{M}_M+\mathscr{M}_B, \tag{3.10}$$

by substituting (2.10), (3.7) and

$$\int_H \hbar\nabla\phi\,.\,\nabla\hat{\phi}\, dA = \mathscr{M}_M, \tag{3.11}$$

which is obtained by setting $\psi = \hat{\phi}$ in (3.2), into the Schwarz inequality

$$\langle \hbar\nabla\phi\,.\,\nabla\hat{\phi}\rangle^2 \leqslant \langle \hbar(\nabla\phi)^2\rangle\langle \hbar(\nabla\hat{\phi})^2\rangle. \tag{3.12}$$

Now suppose that ϕ_* satisfies (2.6)–(2.8) with $\hbar$ replaced by $\hbar_*$ therein. It then follows from (2.10) and (2.11) that the corresponding inertial parameter is given by (note that the functional $\mathfrak{M}_M$ does not, whereas $\mathfrak{M}_H$ does, involve $\hbar$ explicitly)

$$\mathscr{M}_* \equiv \int_H \hbar_*(\nabla\phi_*)^2 dA = \mathfrak{M}_M(\phi_*) \tag{3.13}$$

and from (3.5) and (3.6) that

$$\mathfrak{M}_<(\phi_*) = 2\mathscr{M}_*-\mathfrak{M}_H(\phi_*) \leqslant \mathscr{M}. \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$\mathscr{M}-\mathscr{M}_* > \int_H (\hbar_*-\hbar)(\nabla\phi_*)^2 dA, \tag{3.15}$$

from which it follows that increasing/decreasing $\hbar$ decreases/increases $\mathcal{M}$. This result, which is closely related to the result that increasing/decreasing $\hbar$ increases/decreases the lowest resonant frequency in a closed basin (Troesch 1960, p. 279), may be used to bound $\mathcal{M}$ for a harbour of variable depth by comparing it with the results for harbours having uniform depths equal to the maximum and minimum depths in H or, more generally, with the results for harbours having bottoms that bound the actual bottom.

We obtain a somewhat sharper lower bound,

$$\mathcal{M} \geqslant \mathcal{M}_*^2/\mathfrak{M}_H(\phi_*) = \mathcal{M}_*^2\left\{\mathcal{M}_* + \int_H (\hbar - \hbar_*)(\nabla\phi_*)^2\, dA\right\}^{-1}, \tag{3.16}$$

by substituting (2.10), (3.13) and

$$\int_H \hbar\nabla\phi\,.\,\nabla\phi_*\, dA = \mathfrak{M}_M(\phi_*) = \mathcal{M}_*, \tag{3.17}$$

which is obtained by setting $\psi = \phi_*$ in (3.2), into the Schwarz inequality obtained by replacing $\hat{\phi}$ by ϕ_* in (3.12).

Combining the derivations of (3.9) and (3.15), we obtain

$$\mathcal{M} > \mathcal{M}_{*M} + \mathcal{M}_{*B} + \int_A (\hbar_* - \hbar)(\nabla\hat{\phi}_*)^2\, dA, \tag{3.18}$$

where $\hat{\phi}_*$ satisfies (2.6), (2.7*b*) and (2.8), and $\mathcal{M}_{*M}$ and $\mathcal{M}_{*B}$ are given by (3.8*a, b*) with $\hbar$ replaced by $\hbar_*$ in (3.8*b*).

4. Uniform depth

The assumption $\hbar = 1$ reduces (2.6) to Poisson's equation,

$$\nabla^2\phi = -1/A \quad (\hbar \equiv 1). \tag{4.1}$$

A solution of (4.1) that satisfies (2.7*b*) and (2.8) is given by

$$\phi(\mathbf{r}) = \tfrac{1}{4}A^{-1}(\langle r^2\rangle - r^2) + \phi_M(\mathbf{r}) - \langle\phi_M\rangle, \tag{4.2}$$

where $r = |\mathbf{r}|$ is the radius from the midpoint of M, $-\frac{1}{4}r^2/A$ is a particular solution of (4.1) that yields $\phi_x = 0$ on $x = 0$, and

$$\pi\phi_M(\mathbf{r}) = \int_M f(\eta)\log\{4|\mathbf{r}-(0,\eta)|/a\}\, d\eta \tag{4.3a}$$

$$\sim \log(4r/a) + O(a^2/r^2) \quad (r/a \to \infty) \tag{4.3b}$$

represents a potential flow (solution of Laplace's equation) that satisfies (2.7*b*) and yields $\phi_x = 0$ on $x = 0$, $|y| > \frac{1}{2}a$. Substituting (4.3*b*) into (4.2) and taking the gradient yields

$$\nabla\phi \sim \mathbf{r}\{(\pi r^2)^{-1} - (2A)^{-1}\} \tag{4.4a}$$

$$= 0 \quad \text{on} \quad r = (2A/\pi)^{\frac{1}{2}} \equiv 2^{\frac{1}{2}}R. \tag{4.4b}$$

Accordingly, (4.2) provides the solution of (2.6)–(2.8) for a semicircular harbour of depth d and radius $2^{\frac{1}{2}}R \gg a$ ($\alpha \ll 1$). It also provides an approximation of the type $\oint$ in §3 for other harbours. Exact solutions of (2.6)–(2.8) for other harbours of uniform depth may be obtained by invoking standard techniques in potential theory but do not appear to be worth pursuing here.

Substituting (4.2) into (3.8*a*), invoking (2.4*b*) and (4.3*b*), and letting $\alpha \downarrow 0$ [see (1.1)] yields

$$\mathscr{M}_M = \langle \pi^{-1} \log (4r/a) - \tfrac{1}{4}(r^2/A) \rangle + F, \tag{4.5a}$$

where

$$\pi F = -\int_M \int_M f(y) f(\eta) \log (4|y-\eta|/a)\, d\eta\, dy. \tag{4.5b}$$

It may be shown that F is non-negative. The approximations (M71 §3)

$$f^{(a)}(y) = 1/a, \quad f^{(b)}(y) = \pi^{-1}\{(\tfrac{1}{2}a)^2 - y^2\}^{-\frac{1}{2}}, \tag{4.6a,b}$$

which correspond to (*a*) plane-wave motion in a channel of width a and (*b*) potential flow through an aperture in an infinite plane, yield

$$F^{(a)} = \pi^{-1}(\tfrac{3}{2} - \log 4) = 0{\cdot}036, \quad F^{(b)} = 0. \tag{4.7a,b}$$

We use $F^{(b)} = 0$ in the subsequent examples, although $F^{(a)}$ may be more accurate for a harbour with an entry channel of length greater than its width. The difference, 0·036, would alter the calculated resonant frequency for a typical harbour by less than 1 %; it also provides a measure of the uncertainty in our approximations to $\mathscr{M}$.

Substituting (4.2) into (3.8*b*), invoking (4.3*b*),

$$\int_B (\;)(\mathbf{n}\,.\,\mathbf{r})\, dl = \int_B (\;)\, r^2\, d\theta \tag{4.8}$$

and

$$\int_B (1, r^2, r^4)\, d\theta = (\pi, 2A, 4A\langle r^2 \rangle), \tag{4.9}$$

where θ is the angle shown in figure 1, and letting $\alpha \downarrow 0$ yields

$$\mathscr{M}_B = (2\pi)^{-1} - \tfrac{1}{2}\langle r^2/A \rangle - \pi^{-1} \int_B (\log r)\{\pi^{-1} - \tfrac{1}{2}(r^2/A)\}\, d\theta. \tag{4.10}$$

Applying (4.5) to the semicircular harbour bounded by $r = 2^{\frac{1}{2}}R$, for which $\mathscr{M}_B = 0$, yields†

$$\mathscr{M} = \mathscr{M}_M = \pi^{-1} \log (4R/a) - 0{\cdot}128. \tag{4.11}$$

Applying (4.5) and (4.10) to a circular harbour of radius R yields

$$\mathscr{M}_M = \pi^{-1} \log (4R/a) - 0{\cdot}119, \quad \mathscr{M}_B = 0{\cdot}080. \tag{4.12a,b}$$

The corresponding value of $\mathscr{M}$ determined by the solution of (2.6)–(2.8) and (4.7*b*), M71 §6, exceeds $\mathscr{M}_M + \mathscr{M}_B$ by 0·080, which would alter the calculated resonant frequency for a harbour with $R = 10a$ and an entry channel of length a by less than 2 %. The two lower bounds given by (3.10) differ by 0·6 % for $R = 10a$.

This last example suggests that (4.5) plus (4.10), or even (4.5) alone, should provide an adequate approximation to $\mathscr{M}$ if B runs along, or close to, $x = 0$

† It can be demonstrated that, of all narrow-mouthed harbours of equal area, the semicircular harbour gives the minimum value of $\mathscr{M}$ (Lee 1974).

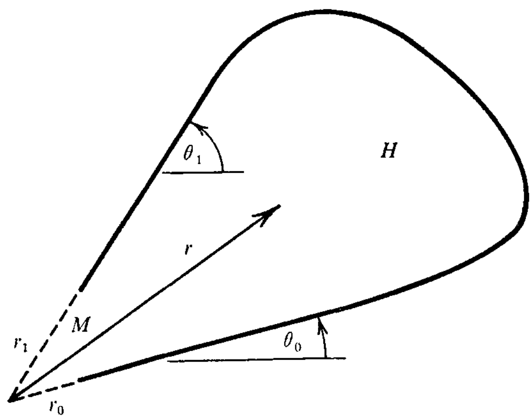

FIGURE 3. Schematic diagram of harbour for which B and M intersect at angles $\frac{1}{2}\pi - \theta_1$ and $\frac{1}{2}\pi + \theta_0$.

for $|y| < y_1 \ll a$. On the other hand, the accuracy deteriorates as M approaches a sharp corner in H owing to the additional constriction of the flow. We therefore consider the configuration shown in figure 3, in which those segments of B that intersect the ends of M are straight over distances that are large compared with a. Measuring r from the intersection of these two segments, such that they are given by $r > r_{0,1}$ on $\theta = \theta_{0,1}$, $\theta_1 > \theta_0$ (see figure 3), and mapping the sector $\theta_0 \leqslant \theta \leqslant \theta_1$ onto a half-plane through the transformation

$$se^{i\chi} = (r/r_1)^\nu e^{i\nu(\theta-\theta_0)}, \quad \nu = \pi/(\theta_1 - \theta_0), \tag{4.13a,b}$$

we pose ϕ_M in the form

$$\phi_M(\mathbf{r}) = \pi^{-1}\int_{M_\nu} f_\nu(\xi)\log\{2^\nu|\mathbf{s}-(0,\xi)|\}\,d\xi \tag{4.14a}$$

$$\sim (\nu/\pi)\log(2r/r_1) \quad (r/a \to \infty), \tag{4.14b}$$

where M_ν is the projection of M on the real axis of the $\mathbf{s}$ plane and f_ν is the normal derivative of ϕ_M in M_ν. Setting $\theta_0 = -\frac{1}{2}\pi$, $\theta_1 = \frac{1}{2}\pi$, $\nu = 1$, $r_1 = \frac{1}{2}a$, $\xi = 2\eta/a$ and $f_\nu = \frac{1}{2}af$ reduces (4.14) to (4.3).

Substituting (4.14) into (4.2) and calculating $\mathscr{M}_M$ and $\mathscr{M}_B$ as above yields

$$\mathscr{M}_M = \langle(\nu/\pi)\log(2r/r_1) - \tfrac{1}{4}(r^2/A)\rangle + F_\nu \tag{4.15}$$

and
$$\mathscr{M}_B = \tfrac{1}{2}(\nu/\pi) - \tfrac{1}{2}\langle r^2/A\rangle - (\nu/\pi)\int_B (\log r)\{(\nu/\pi) - \tfrac{1}{2}(r^2/A)\}\,d\theta, \tag{4.16}$$

where
$$F_\nu = -\int_M \phi_M f\,dy. \tag{4.17}$$

The explicit calculation of F_ν for an assumed form of $f(y)$ is quite involved for arbitrary ν; however, guided by (4.7b) and the corresponding result for $\nu = 2$ (see below), we replace the specification of $f(y)$, which now enters the calculation only through F_ν, by $F_\nu \doteqdot 0$.

Setting $\theta_0 = 0$, $\theta_1 = \frac{1}{2}\pi$, $r_0 = 0$, $r_1 = a$ and $\nu = 2$ yields a harbour with M adjacent to a rectangular corner (figure 4). In this case, the assumption that

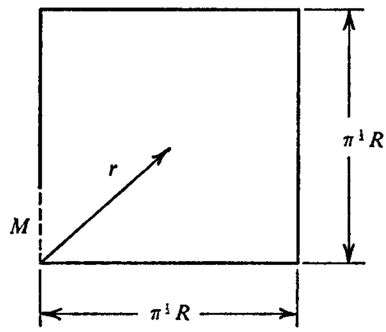

FIGURE 4. Rectangular harbour with mouth at one corner.

$f(y)$ corresponds to potential flow through a gap of width a in the plane $x = 0$, corresponding to M plus its image in the adjacent boundary, leads to $F_\nu = 0$. Applying (4.15) and (4.16) to a square harbour of side $\pi^{\frac{1}{2}}R$ then yields

$$\mathscr{M}_M = (2/\pi)\log(2R/a) - 0{\cdot}037, \quad \mathscr{M}_B = -0{\cdot}001. \qquad (4.18a, b)$$

The value of $\mathscr{M}$ determined by the corresponding solution of (2.6)–(2.8), M71 §7, exceeds $\mathscr{M}_M + \mathscr{M}_B$ by 0·009. The relatively small values of $\mathscr{M}_B$ and the error term in this example reflect the satisfaction of (2.7a) on the two boundaries of H that are adjacent to M.

We now apply the preceding results to the model of Long Beach Harbor previously studied by Lee (1971). The plan is shown in figure 5, and the various parameters are listed in table 1 (some of the data in figure 5 and table 1 refer to §6, where the same example is reconsidered as a pair of coupled basins); the depth is constant ($h \equiv d$) and equal to the outer depth ($d_E = d$). We approximate the entry by a short channel that terminates in the mouth M_1 in the plane of the upper, outer (with reference to figure 5) boundary. $\mathscr{M}_l, \mathscr{M}_M$ and $\mathscr{M}_B$, as calculated from (1.12), (4.15) and (4.16), are listed in table 1. Substituting these data, together with $\mathscr{M}_E$ from (1.10), into (1.14), we obtain $k_0R = 0{\cdot}386$ ($\epsilon = 0{\cdot}468$) for the resonant wavenumber, which agrees with Lee's result $k_0R = 0{\cdot}38$ (both measured and calculated by numerical integration of the full Helmholtz equation).

The preceding examples suggest that the approximation provided by (3.9), (4.15) and (4.16) with $F_\nu = 0$ should provide an adequate approximation to $\mathscr{M}$ for most harbours of uniform depth if $\epsilon \lesssim \frac{1}{2}$.

5. Non-uniform depth

We first recall (see last paragraph in §3) that lower and upper bounds to $\mathscr{M}$ may be obtained by replacing $\hbar$ by its maximum and minimum values. We also recall that the kinetic energy is concentrated in M, so that a good approximation to $\mathscr{M}$ sometimes may be obtained simply by replacing $\hbar$ by its mean value in M, $\hbar_M \equiv 1$ (if d is the mean depth in M).

It does not appear possible to obtain useful analytical solutions of (2.6)

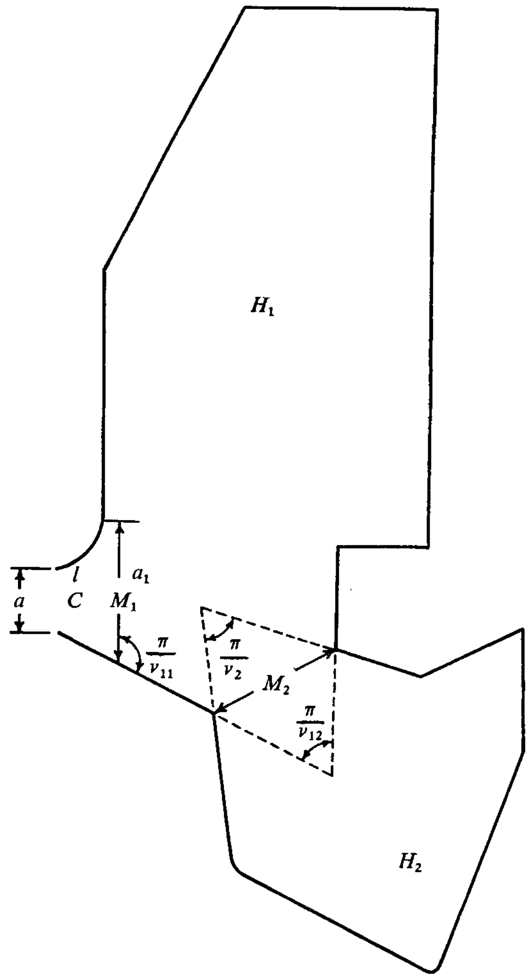

FIGURE 5. Long Beach Harbor (see §§ 4 and 6). The parameters are listed in table 1.

unless the form of h permits separation of variables. If h can be approximated by $h = h(r)$, (2.6) admits the solutions

$$\phi_0 = \int_{\frac{1}{2}r_1}^{r} (\pi r h)^{-1}\, dr, \quad \phi_A = -\tfrac{1}{2}A^{-1}\int_0^r (r/h)\, dr \tag{5.1a, b}$$

as counterparts of $\pi^{-1}\log(2r/r_1)$ and $-r^2/4A$. The resulting counterparts of (4.15) and (4.16) with $F_\nu = 0$ (the variation of h over M being neglected) are

$$\mathscr{M}_M = \langle \nu\phi_0 + \phi_A \rangle \tag{5.2}$$

and

$$\mathscr{M}_B = -\int_B h(\nu\phi_0 + \phi_A)\{(\nu/\pi) - \tfrac{1}{2}(r^2/A)\}\, d\theta. \tag{5.3}$$

If h depends on both r and θ, the solutions (5.1a, b) could be improved by iteration by transferring $r^{-2}(h\phi_\theta)_\theta$ to the right-hand side of (2.6); however, the following examples suggest that the adequacy of the lower bound provided by (3.15) renders such a procedure unprofitable.

$l \equiv$ length of channel $= 0{\cdot}7$
$A_1 \equiv$ area of $H_1 = 45{\cdot}60$, $A_2 \equiv$ area of $H_2 = 17{\cdot}73$
$A \equiv$ total area of harbour $= 63{\cdot}82 = \pi R^2$
$a_1 = 2{\cdot}35$, $a_2 = 2{\cdot}1$

	Single-basin theory H_1+H_2, excitation through M_1	Coupled-basin theory: Inner basin, H_2 excitation at M_2	Coupled-basin theory: Outer basin, H_1 excitation at M_1	Coupled-basin theory: Outer basin, H_1 excitation at M_2
r_1 (cf. figure 3)	2·35	2·15	2·35	2·15
ν (cf. figure 3)	1·552	2·687 $(=\nu_2)$	1·552 $(=\nu_{11})$	2·857 $(=\nu_{12})$
$\mathscr{M}_M$	0·377	0·845	0·490	1·212
$\mathscr{M}_B$	0·043	0·042	0·068	0·302
$\mathscr{M}_M+\mathscr{M}_B$	0·420	0·887	0·558	1·514
$\mathscr{M}_M^2/(\mathscr{M}_M-\mathscr{M}_B)$	0·425	0·889 $=\mathscr{M}_2$	0·567 $=\mathscr{M}_{11}$	1·614 $=\mathscr{M}_{22}$
$\mathscr{M}_{12}$	—	0·722		
$\mathscr{M}_l$	1·559	1·559		
k_1R	0·386	0·389		
k_2R	—	0·798		

TABLE 1. Long Beach Harbor (unit of length based on mouth width a of figure 5)

Applying (5.1)–(5.3) to the semicircular planform of §4 (for which $\nu = 1$ and $r_1 = \frac{1}{2}a$) with the paraboloidal bottom

$$h = 1-\tfrac{1}{2}(1-h_1)(r/R)^2 \quad (0 \leqslant r \leqslant 2^{\frac{1}{2}}R) \tag{5.4}$$

and letting $a/R \downarrow 0$, we obtain

$$\mathscr{M} = \mathscr{M}(4.11)+(2\pi)^{-1}\{\tfrac{3}{2}-(1-h_1)^{-1}-h_1^2(1-h_1)^{-2}\log h_1\}, \tag{5.5}$$

where $\mathscr{M}(4.11)$ is the corresponding result for $h = 1$. We note that (5.5) is asymptotically $(\alpha, \epsilon \downarrow 0)$ exact in this example. Choosing $h_* = 1$ and $\phi_* = \phi(4.2)$ in (3.15) yields the lower bound

$$\mathscr{M} > \mathscr{M}(4.11)+0{\cdot}053(1-h_1). \tag{5.6}$$

The corresponding results for the conical bottom

$$h = 1-2^{-\frac{1}{2}}(1-h_1)(r/R) \tag{5.7}$$

are
$$\mathscr{M} = \mathscr{M}(4.11)+\pi^{-1}[\tfrac{3}{4}-(1-h_1)^{-3}-\tfrac{1}{2}(1-h_1)^{-2}+\tfrac{5}{3}(1-h_1)^{-1} - \{(1-h_1)^{-2}-1\}^2\log h_1] \tag{5.8}$$
and
$$\mathscr{M} > \mathscr{M}(4.11)+0{\cdot}170(1-h_1). \tag{5.9}$$

The results (5.5), (5.6), (5.8) and (5.9) are plotted in figure 6. We remark that the increase in $\mathscr{M}$ associated with the paraboloidal decrease in depth for $h_1 > \frac{3}{8}$ is less than the uncertainty associated with the choice of $f(y)$, roughly 0·04. The increase in $\mathscr{M}$ associated with a conical decrease in depth is appreciably larger (since the change in depth in the neighbourhood of M is relatively larger), but is less than 0·1 for $h_1 > \frac{1}{2}$.

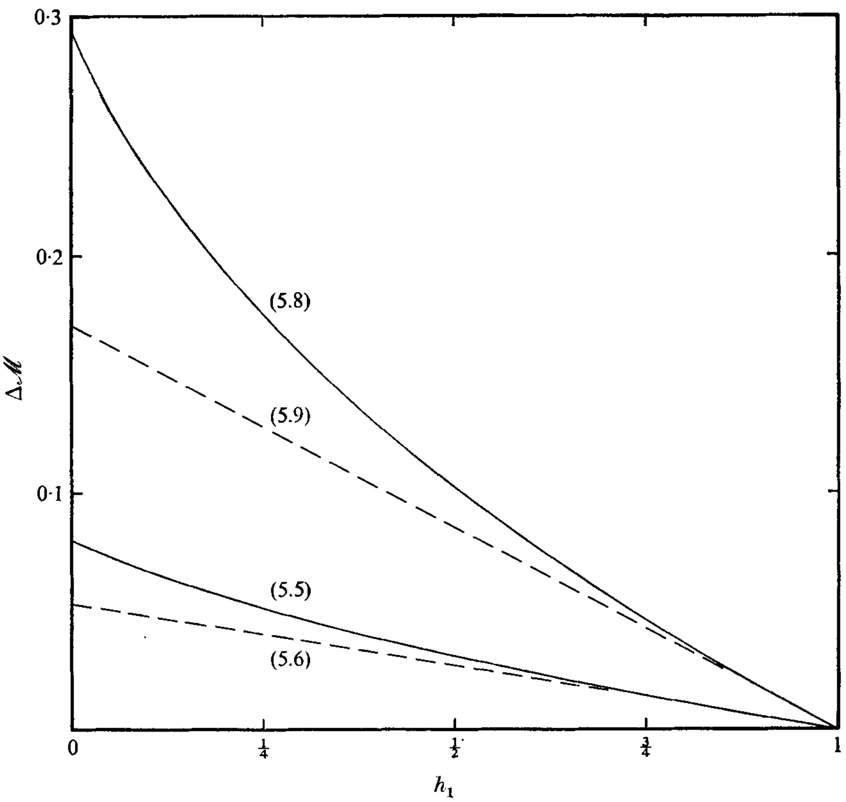

FIGURE 6. The increase in $\mathscr{M}$ associated with a decrease in depth in a semicircular harbour. The bottom is paraboloidal (5.4) for the bottom two curves and conical for the upper two curves. ——, exact results for $\alpha, \epsilon \downarrow 0$; ––––, based on the approximation (3.15).

We surmise from these examples that (3.15) should provide an adequate estimate of $\mathscr{M}$ for most harbours. Moreover, the examples of §4 suggest that (3.18) also should provide an adequate estimate if $1-\hbar = O(r)$ as $r \downarrow 0$. Proceeding on this hypothesis and approximating $\hat{\phi}_*$ by (4.2) and (4.14), we obtain

$$\mathscr{M} > \mathscr{M}_M(4.15) + \mathscr{M}_B(4.16) + \int_H \left(\frac{1-\hbar}{r}\right)\left(\frac{\nu}{\pi} - \frac{r^2}{2A}\right)^2 \frac{dA}{r}. \tag{5.10}$$

6. Coupled basins

We now generalize the preceding formulation to a harbour (see, for example, figure 5) that comprises two basins H_1 and H_2 mutually coupled through M_2 and coupled to the exterior through M_1. Both M_1 and M_2 are assumed to be narrow in the sense of (1.1), whilst the counterpart of (1.2) is

$$\epsilon_i \equiv \omega^2 A_i/gd \ll 1, \tag{6.1}$$

where A_i is the free-surface area of H_i and d is the reference depth.

Let $I_i f_i(s)$ be the volumetric flux per unit width and I_i the total volumetric flux through M_i, where f_i is normalized as in (2.4b) and s is the counterpart of y in M_i, and let

$$V_i = \int_{M_i} \zeta f_i ds \doteqdot \zeta_{M_i} \tag{6.2}$$

be the weighted average of ζ in M_i. Linearity then implies the existence of an impedance matrix $\{Z_{ij}\}$ and an impedance Z_2 such that

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ -I_2 \end{pmatrix} \tag{6.3a}$$

and
$$V_2 = Z_2 I_2, \tag{6.3b}$$

whilst conservation of mass implies

$$i\omega A_1 \langle\zeta\rangle_1 = I_1 - I_2, \quad i\omega A_2 \langle\zeta\rangle_2 = I_2 \tag{6.4a, b}$$

for the average displacements $\langle\zeta\rangle_1$ and $\langle\zeta\rangle_2$ in H_1 and H_2. The corresponding equivalent circuit is shown in figure 2(c).

The impedance Z_2 may be constructed as in §§1–5 and is given by

$$Z_2 = (i\omega C_2)^{-1} + i\omega L_2 = (i\omega A_2)^{-1}(1 - \epsilon_2 \mathscr{M}_2), \tag{6.5}$$

where
$$\mathscr{M}_2 = A_2 \langle \hbar (\nabla\phi_2)^2 \rangle_2 \tag{6.6}$$

and ϕ_2 is determined by (2.6)–(2.8) with the subscript 2 appended to each of A, B, H, M and f.

To construct Z_{ij}, we let

$$\zeta = \langle\zeta\rangle_1 \left\{ 1 + \epsilon_1 \left(\frac{I_1\phi_{11} - I_2\phi_{12}}{I_1 - I_2} \right) \right\}, \tag{6.7}$$

where
$$\nabla \cdot (\hbar \nabla \phi_{1i}) = -1/A_1, \quad \langle\phi_{1i}\rangle_1 = 0, \tag{6.8a, b}$$

and
$$\hbar(\partial\phi_{1i}/\partial n) = \begin{cases} 0 & \text{on} \quad B_1, & (6.9a) \\ -f_i(s) & \text{on} \quad M_i. & (6.9b) \end{cases}$$

Proceeding as in §§1–3, we obtain

$$Z_{ij} = (i\omega C_1)^{-1} + i\omega L_{ij} = (i\omega A_1)^{-1}(1 - \epsilon_1 \mathscr{M}_{ij}), \tag{6.10}$$

where
$$\mathscr{M}_{ij} = \mathscr{M}_{ji} = A_1 \langle \hbar \nabla\phi_{1i} \cdot \nabla\phi_{1j} \rangle \tag{6.11a}$$

$$= -\int_{M_j} \phi_{1i} f_j ds = -\int_{M_i} \phi_{1j} f_i ds. \tag{6.11b}$$

The ϕ_{1i} and ϕ_2 may be estimated as in §§4 and 5. We consider, for example, the model of Long Beach Harbor (figure 5), already considered as a single basin in §4. The angles π/ν_{11}, π/ν_{12} and π/ν_2 are defined according to (4.13b). $\mathscr{M}_H$ and $\mathscr{M}_B$ for H_2 and for H_1 excited through either M_1 or M_2 are calculated from (4.15) and (4.16) with $F_\nu = 0$; these results are used to calculate $\mathscr{M}_2$ (H_2 excited through M_2), $\mathscr{M}_{11}$ (H_1 excited through M_1) and $\mathscr{M}_{22}$ (H_1 excited through M_2) using both (3.9) and (3.10). $\mathscr{M}_{12}$ is calculated from (6.11b), using (4.6b) for f_1 and (4.14b) for ϕ_{12}. The corresponding resonant wavenumbers, obtained by requiring the input reactance at E to vanish, are $kR = 0{\cdot}389$ and $0{\cdot}798$. The lower value is less than 1 % above that given by the single-basin approximation of §4 and cannot be distinguished therefrom in comparison with Lee's result: this suggests that the additional complications of the two-basin model, vis-à-vis its single-basin counterpart, may not be worthwhile in many applications. The higher value of kR is 16 % below Lee's result (0·95), but the agreement is perhaps better than might have been expected in view of the fact that the corresponding value of ϵ is 2·0 (this mode, in which the dominant motion is between H_1 and H_2 through M_2, is much less significant for tsunami response than the dominant mode, both because of its higher frequency and because it is less efficiently excited through M_1).

7. Separation loss

Following conventional hydraulic practice, we represent the head loss at the harbour mouth, or at either end of an entrance channel, in the form

$$\Delta\zeta \equiv \Delta\hat{V} = \tfrac{1}{2}C_S g^{-1}|\hat{u}|\hat{u}, \quad \hat{u} = \hat{I}/ad, \tag{7.1a, b}$$

where C_S is an empirical coefficient (see below) and $\hat{u}$ is the spatial average of the velocity through the mouth. Multiplying this head loss by the flow rate $\hat{I}$ and taking the temporal mean (which $\langle\ \rangle$ now indicates) yields the specific dissipation rate

$$D_S = \langle \hat{I}\Delta\hat{V}\rangle = \tfrac{1}{2}C_S g^{-1} ad\langle|\hat{u}|^3\rangle = (2C_S/3\pi)\, g^{-1} ad|u|^3, \tag{7.2a}$$

where

$$u = I/ad. \tag{7.2b}$$

Substituting (7.2a, b) into (1.17) yields

$$R_S = (4C_S/3\pi)\,(gad)^{-1}|u|. \tag{7.3}$$

A convenient reference value for R_S is the radiation resistance for an ocean of depth d,

$$R_E^{(d)} = \tfrac{1}{2}\omega(gd)^{-1}. \tag{7.4}$$

Dividing (7.3) by (7.4) yields

$$R_S/R_E^{(d)} = (8C_S/3\pi)\,\mathscr{U}, \tag{7.5a}$$

where

$$\mathscr{U} = |u|/\omega a \tag{7.5b}$$

is an inverse Strouhal number for the mouth (but note that the conventional Strouhal number is referred to $\omega/2\pi$ rather than ω).

Dimensional analysis, together with experimental results for the related problems of oscillating flat plates (Keulegan & Carpenter 1958) and acoustical orifices (Ingard & Ising 1967), suggest that, in general,

$$C_S = C_S(Re, \mathscr{U}) \quad (Re = |u|a/\nu) \tag{7.6}$$

but that the dependence on Reynolds number Re should be unimportant in the present context, in which $Re \gg 1$ and the separated flow is assumed to be fully turbulent. The acoustical experiments suggest that C_S is an increasing function of $\mathscr{U}$ but are not quantitatively applicable in the present context owing to the geometrical differences. Elementary considerations (as in one-dimensional hydraulics) suggest that the asymptotic value ($\mathscr{U} \gg 1$) of C_S for an aperture should be the sum of the commonly accepted loss coefficients for contraction and expansion of a channel, 1·0 + 0·5 = 1·5 (the limiting value of C_S reported by Ingard & Ising for a circular acoustical orifice is 2·0), and Ito (1970) reports that the use of this value in a study of breakwaters at tsunami frequencies yields results in agreement with field measurements.† Similarly, the limiting value of C_S at either the entrance or the exit of a channel should be roughly 0·75, the average of the steady-flow contraction and expansion coefficients.

Ito gives results for a numerical model of Ofunato Harbour with and without a breakwater. The dominant mode in the absence of the breakwater has a period of 37 min and a Q (estimated from the half-power bandwidth) of roughly 4. Using his data, $d = 17$ m, $a = 200$ m, $b = 1700$ m (width of exterior channel) and $u \doteq 3$ m/s, yields $\mathscr{U} = 7-8$ and $R_S/R_E = 10$, which implies $Q = 0{\cdot}4$; this is consistent with his result that the resonant response of the dominant mode is completely suppressed by the breakwater. [The dominant mode in Ofunato Harbour without the breakwater corresponds roughly to a quarter-wave, open-organ-pipe-like resonance. The resonance with the breakwater would be qualitatively of the Helmholtz type were it not for the separation losses; using the results of Miles & Munk (1961) for an equivalent rectangular harbour radiating into a semi-infinite sea through a narrow aperture with no other losses yields a period of 49 min and a Q of 6.]

8. Bottom friction

Let c_f be an empirically determined friction coefficient, such that the magnitude of the shear stress on the bottom is given by

$$\tau = c_f \rho \acute{q}^2, \tag{8.1}$$

where $\hat{q}$ is the particle velocity just outside the boundary layer on the harbour bottom. The specific dissiption rate then is given by

$$D_B = g^{-1} c_f \iint \langle \hat{q}^3 \rangle \, dA = \tfrac{4}{3}(\pi g)^{-1} c_f \iint |q|^3 \, dA, \tag{8.2}$$

where the integrals are over the harbour bottom. Invoking (1.17) yields

$$R_B = \tfrac{8}{3}(\pi g |I|^2)^{-1} c_f \iint |q|^3 \, dA. \tag{8.3}$$

The horizontal motion in the Helmholtz mode is concentrated in the neighbourhood of the mouth; accordingly, we obtain a first approximation to q by invoking the known solution for potential flow through the gap $|y| < \frac{1}{2}a$ in the plane $x = 0$:

$$|q| = (2|u|/\pi)(\cos^2 \xi - \cosh^2 \eta)^{-\frac{1}{2}}, \tag{8.4}$$

where $$x = \tfrac{1}{2} a \sinh \xi \sin \eta, \quad y = \tfrac{1}{2} a \cosh \xi \cos \eta, \tag{8.5\,a, b}$$

† The Reynolds numbers for typical laboratory models, such as that of Long Beach Harbor (see above), are too low to test the results of this and the following section.

and $|u|$ is the mean velocity through the gap. Substituting (8.4) and (8.5) into (8.3) and integrating over the half-plane (which is asymptotically equivalent to integrating over A in the limit $\alpha \downarrow 0$), we obtain

$$\iint |q|^3 dA = (2|u|/\pi)^3 (\tfrac{1}{2}a)^2 \int_0^\infty d\xi \int_0^\pi (\cosh^2\xi - \cos^2\eta)^{-\frac{1}{2}} d\eta \qquad (8.6a)$$

$$= 0{\cdot}443a^2|u|^3, \qquad (8.6b)$$

where (8.6*b*) follows from (8.6*a*) with aid of known results for elliptic integrals.

Substituting (7.2*b*) and (8.6*b*) into (8.3), and dividing the result by (7.3), we obtain

$$R_B = 0{\cdot}376c_f(gd^2)^{-1}|u| \qquad (8.7)$$

and

$$R_B/R_S = 0{\cdot}886(c_f/C_S)(a/d). \qquad (8.8)$$

We emphasize that (8.7) and (8.8) rest on the hypothesis that the horizontal motion is concentrated in the neighbourhood of the harbour mouth and therefore are expected to be valid only for $\alpha \ll 1$ [see (1.1)].

The effective Strouhal number for c_f now must be based on the boundary-layer thickness, rather than a as in (7.5); accordingly, it is quite small for tsunamis, and we may safely use that value of c_f which would be appropriate for tidal friction over the same bottom. The most frequently cited value is $c_f = 2 \times 10^{-3}$ (e.g., Taylor 1919), although values as high as 10^{-2} have been reported (Putnam & Johnson 1949; Hasselmann & Collins 1968). Substituting the former value, together with $C_S = 1{\cdot}5$, into (8.8) yields

$$R_B/R_S \doteqdot 1{\cdot}2 \times 10^{-3}(a/d), \qquad (8.9)$$

which is small for any narrow-mouthed harbour for which Helmholtz resonance is likely to be significant (and this statement remains true even if $c_f = 10^{-2}$).

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